Measures of maximal entropy for suspension flows

Tamara Kucherenko, CCNY
(joint work with Dan Thompson)

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Motivation

Bowen and Ruelle, '75

For a suspension flow over the full shift on a finite alphabet with Hölder continuous roof function the measure of maximal entropy is unique and fully supported.

Question: What happens if the roof function is merely continuous?

We obtain that the following possibilities are all realised in the class of suspension flows over the full shift on a finite alphabet with a continuous roof function.

- the measure of maximal entropy is unique but not fully supported
- there are two distinct ergodic measures of maximal entropy with the same support
- the flow may have any prescribed finite number of ergodic measures of maximal entropy
- the number of ergodic measures of maximal entropy is countably infinite
- the set of ergodic measures of maximal entropy is uncountable

Suspension flows

Consider dynamical system (X, f) where $f : X \to X$ is continuous and X is compact. Let $\rho : X \to \mathbb{R}$ be a continuous strictly positive function.

We define the suspension space (relative to ρ) as

$$X_\rho = \{(x, s) : \quad x \in X,\ 0 \le s \le \rho(x)\},$$

where we identify $(x, \rho(x)) = (f(x), 0)$.

The suspension flow with roof function ρ is the flow $(\varphi_t)_{t\ge 0}$ on X_ρ defined by

$$\varphi_t(x, s) = (x, s + t) \quad \text{whenever} \quad s + t \in [0, \rho(x)].$$

Suspension flows

Let μ be an f-invariant probability measure on (X, f).

We can lift μ to an φ-invariant probability measure $\tilde{\mu}$ on X_ρ by taking a direct product with the Lebesque measure (normalized).

Moreover, every φ-invariant probability measure on X_ρ can be obtained in this way from an f-invariant probability measure on X and

$$\mu_1 = \mu_2 \iff \tilde{\mu}_1 = \tilde{\mu}_2.$$

We also have

Abramov's formula: $h_{\tilde{\mu}}(\varphi) = \dfrac{h_\mu(f)}{\int \rho \, d\mu}$

Suspension flows

Consider the topological pressure P on (X, f). For any continuous $g : X \to \mathbb{R}$

$$P(g) = \sup\left\{h_\nu(f) + \int g\,d\nu : \nu \text{ is an f-invariant probability measure}\right\}$$

(Variational Principle)

Since $c \mapsto P(-c\rho)$ is monotonic there exists a constant c such that $P(-c\rho) = 0$.

Suppose μ is an equilibrium state for $-c\rho$, i.e. $0 = P(-c\rho) = h_\mu(f) - c\int \rho\,d\mu$.

For any other f-invariant measure ν we have $c = \dfrac{h_\mu(f)}{\int \rho\,d\mu} \geq \dfrac{h_\nu(f)}{\int \rho\,d\nu}$.

By Abramov's formula $h_{\tilde{\mu}}(\varphi) \geq h_{\tilde{\nu}}(\varphi)$ for any φ-invariant probability measure $\tilde{\nu}$ on X_ρ where equality holds iff ν is an equilibrium state for $-c\rho$.

The measures of maximal entropy for (X_ρ, φ) are precisely the lifts the equilibrium states of $-c\rho$.

General idea

The construction of a suspension flow with a specific behavior of measures of maximal entropy reduces to finding a continuous function $g : X \to \mathbb{R}$ such that

- g is negative and bounded away from zero.
- $P(g) = 0$.
- The set of equilibrium states of g has the desired properties.

Our approach:

1. Pick an invariant $Y \subset X$ with an "interesting" set of measures of maximal entropy.
2. Define g in such a way that those measures are the equilibrium states of g.

Subshifts of finite type

Theorem (K., D. Thompson)

Suppose (X, f) is a full shift on a finite alphabet and $Y \subset X$ is a subshift of finite type with positive entropy $h_{\mathrm{top}}(Y)$. Define

$$g(x) = \begin{cases} -h_{\mathrm{top}}(Y), & \text{if } x \in Y \\ a_j, & \text{if } \mathit{dist}(x, Y) = (1/2)^j \end{cases}$$

where letting B_n be the number of blocks in Y of size n we set

$$a_j = -\frac{1}{n} \log B_n - \frac{c}{\sqrt{j}}, \quad \text{when} \quad \frac{n(n-1)}{2} \leq j < \frac{n(n+1)}{2}.$$

Then the measures of maximal entropy for the suspension flow over (X, f) with the roof function $(-g)$ are exactly the lifts of measures of maximal entropy for Y.

Subshifts of finite type

When Y is transitive, the flow has a unique measure of maximal entropy which is not fully supported.

When Y has several transitive components with the same entropy, the flow has several measures of maximal entropy.

When the alphabet is infinite, examples of suspension flows which do not have a unique maximal entropy measure have been obtained by Iommi, Jordan, Todd (2013-2015)

Our theorem provides the first examples of non-uniqueness in the classical case (finite alphabet and positive roof function).

Beyond subshifts of finite type

Question: Can we extend this result beyond subshifts of finite type?

In the proof of the previous theorem we give an explicit combinatorial construction of the roof function and the structure of subshifts of finite type is heavily used.

The construction already breaks down for the even shift, one of the simplest examples of non-finite type subshifts.

Beyond subshifts of finite type

Theorem (K., Thompson)

Suppose X is a compact metric space and $f : X \to X$ is a homeomorphism such that the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Then for any closed f-invariant subset $Y \subset X$ with $h_{\text{top}}(Y) > 0$ there exists a continuous $\rho : X \mapsto (0, \infty)$ so that the measures of maximal entropy for the suspension flow (X_ρ, φ) are exactly the lifts of the measures of maximal entropy for Y.

The proof relies on the Israel's theorem on approximation of the invariant measures by tangents to the pressure function.

Consequences

Corollary 1

There are examples of suspension flows over the full shift on a finite alphabet which have two distinct ergodic measures of maximal entropy with the same support.

Let X be a the full shift on 4 symbols. Take Y to be the Dyck shift.

The Dyck shift is the shift on the four symbols (,), [,] with the restriction that an opening parenthesis (must be closed by) and an opening bracket [must be closed by].

Krieger (1974): The Dyck shift has exactly two ergodic measures of maximal entropy, and each one is fully supported and Bernoulli.

We obtain a suspension flow over the full shift on 4 symbols with two measures of maximal entropy which have the same support and are products of Bernoulli measure with Lebesgue measure.

Consequences

Corollary 2

For suspension flows over the full shift on a finite alphabet, the set of ergodic measures of maximal entropy can have any finite cardinality or be countably infinite.

Haydn (2013): Let $L \in \mathbb{N}$ and X be a the full shift on $2L$ symbols. There is a topologically mixing subshift $Y \subset X$ of positive topological entropy with L distinct ergodic entropy maximizing measures.

Buzzi (2005): Let X be a the full shift on 3 symbols. Then there is a subshift $Y \subset X$ with $h_{\text{top}}(Y) = \log 2$ which supports countably many ergodic measures with entropy $\log 2$.

Consequences

Corollary 3

There exists a suspension flow over the full shift on a finite alphabet with uncountably many ergodic measures of maximal entropy.

Denker, Grillenberger and Sigmund, *Ergodic Theory on Compact Spaces*:
There exists a minimal subshift Σ_0 with zero entropy and uncountably many ergodic invariant probability measures.

Let $Y = \Sigma_0 \times \{0,1\}^{\mathbb{N}}$. Then $h_{\text{top}}(Y) > 0$ and we can apply our result to obtain a flow with continuous roof function which has uncountably many ergodic measures of maximal entropy.

Thank you!